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Metric and Ricci Tensors for a Certain Class of Space–Times of *D* Type

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Abstract

The class of space–times has been determined at the connection level, assuming the existence of some symmetrical relations between the Ricci rotation coefficients. It has been assumed, for instance, that at least two shear-free congruences of null geodesics exist. We have shown that only *D* type or conformally flat space-times can belong to this class. The theorem has been proved that a system of coordinates exists in which the metric tensor can depend on two coordinates, only. The metric tensor has been determined with an accuracy to two functions, each of which is a function of only one coordinate. Linear, second-order differential expressions have been found for these two functions. They determine the Ricci tensor. Several solutions of the Einstein–Maxwell equations with a cosmological constant are given.

1. *Method and Formalism*

When presenting some of the results and in the calculations the tetrad formalism was used, as described by Debney et al. (1969). This formalism has also been presented in abbreviated form in our previous work, preceding the present paper in this issue (Kowalczyński and Plebański, 1977).

The tetrad field of independent null vectors $e^a{}_\mu (a,_\mu = 1, 2, 3, 4)$ determined in the whole space-time is characterized by the fact that the vectors $e^1{}_\mu$ and $e^2{}_\mu$ are complex and conjugate, and the $e^3{}_\mu$ and $e^4{}_\mu$ ones are real. The metric form is as follows:

$$ds^2 = 2e^1 e^2 + 2e^3 e^4 \tag{1.1}$$

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where

$$e^a \underset{\text{def}}{=} e^a{}_\mu \, dx^\mu \tag{1.2}$$

The convention is kept that Greek letters are suffixes of the tensor formalism and Latin letters and numbers those of the tetrad formalism.

Units have been chosen so that the speed of light and constant of gravitation are equal to unity:

$$c = G = 1 \tag{1.3}$$

and the signature is +++–.

2. *Principal Assumptions and Theorem Regarding the Metric Tensor*

The object of our interest in the present work is a class of such solutions of the Einstein equations of (1.1) shape with differential forms Γ_{ab} given by the first structure (Cartan) formulas, which have the following form in our tetrad $e^a{}_\mu$ system:

$$\Gamma_{42} = Ae^1 - Be^3 = \Omega \, du \tag{2.1a}$$

$$\Gamma_{31} = Ae^2 - Be^4 = \Omega \, dv \tag{2.1b}$$

$$\Gamma_{12} + \Gamma_{34} = C(e^1 - e^2) + D(e^3 - e^4) \tag{2.1c}$$

where A, B, C, D, Ω, u, v are complex functions of the coordinates, and we assume that

$$d \ln \Omega = -C(e^1 + e^2) - D(e^3 + e^4) \tag{2.1d}$$

$$\operatorname{Im}(A + D) = \operatorname{Im}(B + C) = 0 \tag{2.1e}$$

In equation (2.1d) the postulate is concealed that

$$\Omega \neq 0 \tag{2.2}$$

The reason for investigating this class of metrics is the fact that to this class belongs a physically interesting metric which includes seven constants and is a generalization of the Kerr-Newman metric (Plebański and Demiański, 1975).

Let us observe that the relations (2.1a)-(2.1d) are invariant with regard to the simultaneous transposing of the tetrad suffixes $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$.

The following relations result directly from equations (2.1a)-(2.1c):

$$\Gamma_{422} = \Gamma_{424} = \Gamma_{311} = \Gamma_{313} = 0 \tag{2.3a}$$

$$\Gamma_{421} = \Gamma_{312} = A \tag{2.3b}$$

$$\Gamma_{423} = \Gamma_{314} = -B \tag{2.3c}$$

$$\Gamma_{121} + \Gamma_{341} = -\Gamma_{122} - \Gamma_{342} = C \tag{2.3d}$$

$$\Gamma_{123} + \Gamma_{343} = -\Gamma_{124} - \Gamma_{344} = D \tag{2.3e}$$

whereas from (2.1a)-(2.1c) and the second structure (Cartan) formulas it follows that among all the independent tetrad components of the Riemann tensor, R_{abcd}, only the following three, $R_{4231}, R_{1212}+R_{3412}, R_{1234}+R_{3434}$, can be different from zero. We obtain for instance that

$$R_{4231} = -2(AD + BC) \tag{2.4}$$

$$R_{22} = R_{24} = R_{44} = R_{33} = R_{23} = 0 \tag{2.5}$$

$$C^{(1)} = C^{(2)} = C^{(4)} = C^{(5)} = 0 \tag{2.6}$$

$$C^{(3)} = R/6 - 4(AD + BC) \tag{2.7}$$

Equations (2.6) tell us that the class of metrics (2.1) may include metrics describing space-times either conformally flat, if $C^{(3)} = 0$, or of Petrov [2, 2] type, if $C^{(3)} \neq 0$, (type D). Besides, it follows from (2.3a) that the independent vectors $e^{3\mu}$ and $e^{4\mu}$ are by assumption geodesic and shear-free, and from (2.6) we have that these vectors are double Debever-Penrose ones if and only if $C^{(3)} \neq 0$.

Let us now introduce the following auxiliary symbols:

$$E^1 \underset{\text{def}}{=} e^1 + e^2 \tag{2.8a}$$

$$E^2 \underset{\text{def}}{=} i(e^1 - e^2) \tag{2.8b}$$

$$E^3 \underset{\text{def}}{=} e^3 + e^4 \tag{2.8c}$$

$$E^4 \underset{\text{def}}{=} e^3 - e^4 \tag{2.8d}$$

The use of these symbols will be in certain situations more convenient than applying symbols e^a. From (2.8) we get the following relations:

$$ds^2 = 2e^1e^2 + 2e^3e^4 = \tfrac{1}{2}[(E^1)^2 + (E^2)^2 + (E^3)^2 - (E^4)^2] \tag{2.9}$$

$$e^1 \wedge e^2 \wedge e^3 \wedge e^4 = -(i/4)\, E^1 \wedge E^2 \wedge E^3 \wedge E^4 \tag{2.10}$$

It follows from (2.10) that the forms E^i are independent and span the whole space-time. Mention should be made that the numbers i in the E^i symbols are not tetrad suffixes.

We have the following:

Theorem 1. For every metric (2.9), if the relations (2.1) hold, then there exist such a system of real coordinates x, y, z, t and such real functions f, g, k, l, p, q, all independent of the variables z and t (hence able to depend only on variables x, y), that

$$E^1 = 2f\, dx \tag{2.11a}$$

$$E^2 = 2k\, dz + 2l\, dt \tag{2.11b}$$

$$E^3 = 2g\,dy \tag{2.11c}$$

$$E^4 = 2p\,dz + 2q\,dt \tag{2.11d}$$

Of course in view of the fact that the forms E^i are independent and span the whole space-time, we must have

$$f \neq 0, \qquad g \neq 0, \qquad kq - lp \neq 0 \tag{2.11e}$$

The proof of Theorem 1 is long and involves laborious calculations. Darboux theorems are used. The scheme of that proof is given in the Appendix.

In the premise of Theorem 1 the conditions (2.1) can be modified by replacing condition (2.1e) by the condition $A \neq 0$ or $B \neq 0$, but then the range of the theorem will be reduced (see the last paragraph in the Appendix).

3. *Determination of the Metric Tensor and of Ricci Rotation Coefficients in a General Way*

By Theorem 1 the starting point for our further considerations are relations (2.11). Substituting the forms e^a given by (2.8) and (2.11) in the first Cartan formulas we obtain the following expressions for Γ_{abc}:

$$\Gamma_{421} = \Gamma_{312} = A = -\frac{f_{,y}}{4fg} + \frac{1}{4gr}(pl_{,y} - qk_{,y}) + \frac{i}{4fr}(qp_{,x} - pq_{,x}) \tag{3.1a}$$

$$\Gamma_{422} = \Gamma_{311} = -\frac{f_{,y}}{4fg} + \frac{1}{4gr}(qk_{,y} - pl_{,y}) + \frac{i}{4fr}(kl_{,x} - lk_{,x}) \tag{3.1b}$$

$$\Gamma_{423} = \Gamma_{314} = -B = \frac{g_{,x}}{4fg} + \frac{1}{4fr}(kq_{,x} - lp_{,x}) + \frac{i}{4gr}(kl_{,y} - lk_{,y}) \tag{3.1c}$$

$$\Gamma_{424} = \Gamma_{313} = \frac{g_{,x}}{4fg} + \frac{1}{4fr}(lp_{,x} - kq_{,x}) + \frac{i}{4gr}(qp_{,y} - pq_{,y}) \tag{3.1d}$$

$$\Gamma_{121} + \Gamma_{341} = -\Gamma_{122} - \Gamma_{342} = C = \frac{1}{2fr}(pl_{,x} - qk_{,x}) + \frac{i}{4gr} \times (kl_{,y} - lk_{,y} + pq_{,y} - qp_{,y}) \tag{3.1e}$$

$$\Gamma_{123} + \Gamma_{343} = -\Gamma_{124} - \Gamma_{344} = D = \frac{1}{2gr}(lp_{,y} - kq_{,y}) + \frac{i}{4fr}(kl_{,x} - lk_{,x} + pq_{,x} - qp_{,x}) \tag{3.1f}$$

where

$$r \underset{\text{def}}{=} kq - lp \tag{3.1g}$$

Each independent quantity Γ_{abc} was calculated here separately, without making use of relations (2.3). Hence, equations (2.3b)-(2.3e) as well as the corresponding two equalities in (2.3a) (without equalling to zero) result from (2.11). This means that if we assume (2.11), then some of the relations (2.1) are conclusions from other ones.

Considering the last of conditions (2.11e), at least in one of the pairs k, q or l, p both functions must be different from zero. Without loss of generality the choice of such a pair is arbitrary. Let us assume that

$$k \neq 0, \qquad q \neq 0 \tag{3.2}$$

Setting the right-hand sides of (3.1b) and (3.1d) equal to zero on the basis of (2.3a) and using (3.2) we get after integration that $l = Fk$, $p = -Gq$, $r = kq - lp = kq(1 + FG)$, $f = hk(1 + FG)$, $g = jq(1 + FG)$, where the functions

$$F = F(y), \qquad G = G(x) \tag{3.3}$$

and $h = h(x)$, $j = j(y)$ are disposable functions of one variable, x or y, appearing in the course of integration. From the relations just obtained and conditions (2.11e) it follows that

$$1 + FG \neq 0 \tag{3.4}$$

By integrating the real part of equation (2.1d) and making use of (3.2) we come to the conclusion that there exists such a function M for which $k = MP$ and $q = MQ$, where

$$M = M(x, y) \neq 0, \qquad P = P(x) \neq 0, \qquad Q = Q(y) \neq 0 \tag{3.5}$$

The functions P, Q of one variable, x or y, appear as disposable functions in the course of integration.

Let us now perform the following transformation of the coordinates x and y: $dx' = hP^2\, dx$, $dy' = jQ^2\, dy$. After dropping the primes the expressions for E^i and the independent quantities Γ_{abc} assume in the modified x, y, z, t coordinate system the form

$$E^1 = (2M/P)\,(1 + FG)\, dx \tag{3.6a}$$

$$E^2 = 2MP(dz + F\, dt) \tag{3.6b}$$

$$E^3 = (2M/Q)\,(1 + FG)\, dy \tag{3.6c}$$

$$E^4 = 2MQ(dt - G\, dz) \tag{3.6d}$$

$$\Gamma_{421} = \Gamma_{312} = A = -\frac{Q\,\partial_y [M(1 + FG)]}{2M^2(1 + FG)^2} - \frac{iQG_{,x}}{4M(1 + FG)^2} \tag{3.7a}$$

$$\Gamma_{423} = \Gamma_{314} = -B = \frac{P\,\partial_x [M(1 + FG)]}{2M^2(1 + FG)^2} + \frac{iPF_{,y}}{4M(1 + FG)^2} \tag{3.7b}$$

$$\Gamma_{121} + \Gamma_{341} = -\Gamma_{122} - \Gamma_{342} = C = -\frac{\partial_x(MP)}{2M^2(1+FG)} + \frac{iPF_{,y}}{4M(1+FG)^2} \tag{3.7c}$$

$$\Gamma_{123} + \Gamma_{343} = -\Gamma_{124} - \Gamma_{344} = D = -\frac{\partial_y(MQ)}{2M^2(1+FG)} + \frac{iQG_{,x}}{4M(1+FG)^2} \tag{3.7d}$$

The above transformation of the old coordinates x, y is such that neither the factor $[P(x)]^{-1}$ in the form E^1 nor the factor $[Q(y)]^{-1}$ in the form E^3 have been absorbed by the differentials of the new coordinates, dx and dy, respectively. This was done deliberately to simplify the expressions that will be used later.

It is seen from relations (2.9) and (3.6) that the quantity $2M^2$ has the character of a conformal coefficient for all metrics ds^2 belonging to class (2.1). Furthermore, it can be easily observed that (2.1e) results from (3.7). This means that if we make the assumptions (2.1a)-(2.1d) then the relations (2.1e) are equivalent to relations (2.11).

From equations (2.1a), (2.1b), and (2.1d) we can conclude that the Ω function is determined with an accuracy to any arbitrary multiplicative constant. If we assume that $\Omega = |\Omega| e^{i\varphi}$, then $|\Omega|$ and φ are determined with an accuracy to an arbitrary multiplicative and additive constant, respectively. On the basis of the latter and of the result of integration of the real part of equation (2.1d) performed earlier we can assume that

$$|\Omega| = MPQ \tag{3.8}$$

The imaginary part of relation (2.1d) takes the following form:

$$d\varphi = -\tfrac{1}{2}\,\frac{F_{,y}}{1+FG}\,dx - \tfrac{1}{2}\,\frac{G_{,x}}{1+FG}\,dy \tag{3.9}$$

If we subtract equations (2.1a) and (2.1b) from each other, noting that the forms E^2 and E^4 contain only differentials dz and dt and that all the functions that we use here are functions of variables x, y, only, and make use of (3.8), we obtain the following two complex equations:

$$\frac{1}{M^2(1+FG)^2}\left\{-\partial_x[M(1+FG)] + \tfrac{1}{2}MFG_{,x} - iF\,\partial_y[M(1+FG)] - \frac{i}{2}MF_{,y}\right\} = a_1 e^{i\varphi} \tag{3.10a}$$

$$\frac{1}{M^2(1+FG)^2}\left\{-\partial_y[M(1+FG)] + \tfrac{1}{2}MGF_{,y} - iG\,\partial_x[M(1+FG)] - \frac{i}{2}MG_{,x}\right\} = a_2 e^{i\varphi} \tag{3.10b}$$

where a_1, a_2 are complex constants.

It appears that summation of equations (2.1a) and (2.1b), taking account of (3.3), (3.5)-(3.8), and (3.10), gives no relations that could determine more accurately the functions occurring in (3.6).

4. *Determination of the F, G, M Functions and of Expressions for Tetrad Components of the Ricci Tensor*

The considerations in this paragraph will be divided into four separate parts corresponding to the cases when $F_{,y} = G_{,x} = 0$; $F_{,y} = 0$ and $G_{,x} \neq 0$; $F_{,y} \neq 0$ and $G_{,x} = 0$; $F_{,y} \neq 0$ and $G_{,x} \neq 0$.

4.1. *The case* $F_{,y} = G_{,x} = 0$. If

$$F = G = 0 \tag{4.1a}$$

then it follows from (3.9) that $\varphi = \text{const}$, which, in view of the earlier mentioned fact that φ is determined with accuracy to an arbitrary additive constant, authorizes us to assume without loss of generality that

$$\varphi = 0 \tag{4.1b}$$

It then results from equations (3.10) that both constants a_1 and a_2 are real, and that

$$M = (a_1 x + a_2 y + a_3)^{-1} \tag{4.1c}$$

where a_3 is an arbitrary real constant. From (2.9) and (3.6) we see that the shape of form ds^2 will not change if we assume that $a_3 = 0$ in the case when at least one of the constants a_1, a_2 is different from zero. If, on the other hand, $a_1 = a_2 = 0$, then a_3 may have any arbitrary value different from zero, since, as follows from (3.6), the function M is determined with accuracy to the arbitrary multiplicative constant.

If we keep to the general expression for M given by (4.1c) and make use of the second Cartan formulas as well as of the relations (3.7) and (4.1a), we can find the independent quantities R_{ab} missing in (2.5):

$$\begin{aligned} R_{12} = {} & \tfrac{1}{4}(a_1 x + a_2 y + a_3)^2\, \partial_x\, \partial_x P^2 - a_1(a_1 x + a_2 y + a_3)\, \partial_x P^2 \\ & - \tfrac{1}{2} a_2 (a_1 x + a_2 y + a_3)\, \partial_y Q^2 + \tfrac{3}{2} a_1{}^2 P^2 + \tfrac{3}{2} a_2{}^2 Q^2 \end{aligned} \tag{4.1d}$$

$$\begin{aligned} R_{34} = {} & \tfrac{1}{4}(a_1 x + a_2 y + a_3)^2\, \partial_y\, \partial_y Q^2 - a_2(a_1 x + a_2 y + a_3)\, \partial_y Q^2 \\ & - \tfrac{1}{2} a_1 (a_1 x + a_2 y + a_3)\, \partial_x P^2 + \tfrac{3}{2} a_1{}^2 P^2 + \tfrac{3}{2} a_2{}^2 Q^2 \end{aligned} \tag{4.1e}$$

In the situation when one of the constants F, G or both of them are different from zero we can, performing simple linear transformations of the coordinates, bring relations (3.6), and hence also (3.7), to such a form in the new coordinate system that the effect is such as if we assumed in the old coordinate system $F = G = 0$ in equations (3.6). Hence, the relations (4.1) given here are most general for the case $F_{,y} = G_{,x} = 0$.

4.2. *The case* $F_{,y} = 0$ *and* $G_{,x} \neq 0$. If

$$F = 0 \tag{4.2a}$$

then, as in the previous case, because φ is determined with accuracy to any arbitrary additive constant and in view of the fact that the transformation of the shift of the coordinates by a constant has no effect on the relations (3.6), (3.9), and (3.10), we find from (3.9) and (3.10) without loss of generality that

$$\varphi = \omega y, \qquad \omega = \bar{\omega} = \text{const} \neq 0 \tag{4.2b}$$

$$G = -2\omega x \tag{4.2c}$$

$$a_1 = 0, \qquad a_2 = \bar{a}_2 \neq 0 \tag{4.2d}$$

$$M = \frac{\omega}{a_2 \sin(\omega y)} \tag{4.2e}$$

Subsequently, making use of the second Cartan formulas and of the relations (3.7), (4.2a), (4.2c), and (4.2e), we find the missing quantities R_{ab}:

$$R_{12} = \tfrac{1}{4} \frac{a_2{}^2 \sin^2(\omega y)}{\omega^2} \partial_x \partial_x P^2 - \tfrac{1}{2} \frac{a_2{}^2 \sin(\omega y) \cos(\omega y)}{\omega} \partial_y Q^2 \tag{4.2f}$$

$$+ \tfrac{1}{2}(3 \cos^2(\omega y) - \sin^2(\omega y))a_2{}^2 Q^2$$

$$R_{34} = \tfrac{1}{4} \frac{a_2{}^2 \sin^2(\omega y)}{\omega^2} \partial_y \partial_y Q^2 - \frac{a_2{}^2 \sin(\omega y) \cos(\omega y)}{\omega} \partial_y Q^2$$

$$+ \tfrac{3}{2} a_2{}^2 Q^2 \tag{4.2g}$$

In the case when $F \neq 0$, such transformations of the coordinates x, z, t can be made which also produce changes of the functions G, M, P (keeping the coordinate y and function Q intact), and thus the forms E^i in the new coordinate system are such as if we had put in the old coordinate system $F = 0$ and $G = -2\omega x$ in equations (3.6). Thus the relations (4.2) are most general in the case when $F_{,y} = 0$ and $G_{,x} \neq 0$.

4.3. *The case* $F_{,y} \neq 0$ *and* $G_{,x} = 0$. This case is symmetrical to the former case, and, as in that case, the most general relations in the case $F_{,y} \neq 0$ and $G_{,x} = 0$ can be presented in the form

$$G = 0 \tag{4.3a}$$

$$\varphi = \omega x, \qquad \omega = \bar{\omega} = \text{const} \neq 0 \tag{4.3b}$$

$$F = -2\omega y \tag{4.3c}$$

$$a_1 = \bar{a}_1 \neq 0, \qquad a_2 = 0 \tag{4.3d}$$

$$M = \frac{\omega}{a_1 \sin(\omega x)} \tag{4.3e}$$

$$R_{12} = \tfrac{1}{4}\,\frac{a_1^2 \sin^2(\omega x)}{\omega^2}\,\partial_x\,\partial_x P^2 - \frac{a_1^2 \sin(\omega x)\cos(\omega x)}{\omega}\,\partial_x P^2 + \tfrac{3}{2}a_1^2 P^2 \tag{4.3f}$$

$$R_{34} = \tfrac{1}{4}\,\frac{a_1^2 \sin^2(\omega x)}{\omega^2}\,\partial_y\,\partial_y Q^2 - \tfrac{1}{2}\,\frac{a_1^2 \sin(\omega x)\cos(\omega x)}{\omega}\,\partial_x P^2 + \tfrac{1}{2}[3\cos^2(\omega x) - \sin^2(\omega x)]\,a_1^2 P^2 \tag{4.3g}$$

One can easily see that if in relations (4.2) or (4.3) the limiting transition $\omega \to 0$ is made, subcases of relations (4.1) are obtained when $a_1 = a_3 = 0$, $a_2 \neq 0$, or $a_2 = a_3 = 0, a_1 \neq 0$, respectively.

4.4. *The case $F_{,y} \neq 0$ and $G_{,x} \neq 0$.* Analyzing equation (3.9) for this case we find that $(F_{,y})^2 = b_1F^2 + b_2F - b_3$, $(G_{,x})^2 = b_3G^2 + b_2G - b_1$, where b_i are real constants. From further prolonged investigations it follows that we must have $b_2^2 + 4b_1b_3 > 0$. After integrating equation (3.9) and calculating the function M from relations (3.10) (the latter can be done algebraically—without integrating) we obtain functions φ and M as functions of F and G and of the seven real parameters $\mathrm{Re}\,a_1$, $\mathrm{Im}\,a_1$, $\mathrm{Re}\,a_2$, $\mathrm{Im}\,a_2$, b_1, b_2, b_3. A laborious analysis of the relations obtained allows us to eliminate five out of these seven parameters. Finally we arrive at the following relations:

$$(F_{,y})^2 = b_1(F^2 - 1) + b_2F \tag{4.4a}$$

$$(G_{,x})^2 = b_1(G^2 - 1) + b_2G \tag{4.4b}$$

$$\varphi = \tfrac{1}{2}\,\mathrm{arc\,cos}\,\frac{-2b_1(F+G) + b_2(FG-1)}{(1+FG)\,(b_2^2 + 4b_1^2)^{1/2}} \tag{4.4c}$$

$$\begin{aligned} M &= F_{,y}(1+FG)^{-1}\,[F(\sin\varphi + \cos\varphi) + \cos\varphi - \sin\varphi]^{-1} \\ &= G_{,x}(1+FG)^{-1}\,[G(\sin\varphi + \cos\varphi) + \cos\varphi - \sin\varphi]^{-1} \end{aligned} \tag{4.4d}$$

As is seen, in the case $F_{,y} \neq 0$ and $G_{,x} \neq 0$ we cannot have $b_1 = b_2 = 0$. Making use of the above relations, of (3.7) and of second Cartan formulas, we obtain the independent quantities R_{ab} lacking in (2.5):

$$R_{12} = f_1\,\partial_x\,\partial_x P^2 + f_2\,\partial_x P^2 + f_3P^2 + f_4\,\partial_y Q^2 + f_5Q^2 \tag{4.4e}$$

$$R_{34} = g_1\,\partial_y\,\partial_y Q^2 + g_2\,\partial_y Q^2 + g_3Q^2 + g_4\,\partial_x P^2 + g_5P^2 \tag{4.4f}$$

where $f_i = f_i(F, G)$, $g_i = g_i(F, G)$. Since the functions f_i, g_i and M are functions of F and G, only, not related directly to x and y, it would seem purposeful to use F and G as coordinates instead of y and x, respectively, which is of course possible in view of the assumption assumed that $F_{,y} \neq 0$ and $G_{,x} \neq 0$. Such a procedure is, however, inconvenient because of the complicated form of the explicit expression $M(F, G)$ which entails as a further consequence very complicated explicit forms of the expressions $f_i(F, G)$ and $g_i(F, G)$, making them of little use for the calculations of functions $P(G)$ and $Q(F)$.

For these reasons, and primarily because of the small practical usefulness of the expressions $f_i(F, G)$ and $g_i(F, G)$, when treating F, G as coordinates, we have not given the explicit forms of the coefficients $f_i(F, G)$ and $g_i(F, G)$ in equations (4.4e) and (4.4f). In practice it is much more convenient to integrate equations (4.4a) and (4.4b), and subsequently make easy transformations of the coordinates $x(x')$ and $y(y')$ converting the expressions $F(y)$ and $G(x)$ to the simpler $F(y')$ and $G(x')$ ones, and finally using the expressions $M(x', y'), f_i(x', y'), g_i(x', y')$. Of course integration of equations (4.4a) and (4.4b) splits into three separate integrations for three separate cases when $b_1 > 0$, $b_1 = 0$ or $b_1 < 0$. The expressions $F(y')$ and $G(x')$ are then hyperbolic, algebraic, or trigonometric.

In the case $b_1 < 0$ particularly interesting from the physical point of view since it includes the mentioned generalization of the Kerr–Newman metric, the expressions R_{12} and R_{34} have been given in explicit form by Plebański and Demiański (1976) (see also Plebański, 1975a, b).

In this way by (2.9) and (3.6) the class of metrics given by the assumptions (2.1) has been fully determined with an accuracy to two unknown functions $P(x)$ and $Q(y)$, each of one variable. Owing to the determination of the expressions for R_{12} and R_{34} (the remaining R_{ab} being equal to zero identically) the functions P and Q can be determined from the Einstein equations, and hence depend on what energy-momentum tensor $T_{\mu\nu}$ is used in these equations. This tensor can be chosen arbitrarily with the constraint resulting from the identities (2.5) according to which we must have $T_{ab} = 0$ for all pairs of tetrad suffixes (a, b) different from $(1, 2)$ and $(3, 4)$.

It seems that the fact is worth emphasizing that, for the class of metrics we are interested in, only the tetrad components of the Ricci tensor that are different from zero, i.e., R_{12} and R_{34}, may be determined by expressions *linear* with respect to the structural functions P^2, Q^2 and their derivatives.

5. *Several Solutions of Einstein-Maxwell Equations without Currents and Charges but with the Cosmological Constant for the Case when $F_{,y} = G_{,x} = 0$*

Several solutions of the equations

$$\begin{gathered} G_{ab} = -\lambda g_{ab} - 2(F_{ac}F_b{}^c - \tfrac{1}{4}g_{ab}F_{cd}F^{cd}) \\ d(F_{ab}e^a \wedge e^b) = 0, \qquad F^{ab}{}_{;b} = 0 \end{gathered} \tag{5.1}$$

where the quantities F_{ab} are tetrad components of the electromagnetic field tensor $F_{\mu\nu}$, and λ is the cosmological constant, have been obtained in the form (2.9) and (3.6) if the relations (4.1) are fulfilled. These solutions are

$$\begin{gathered} M = 1/y, \qquad P = (ax^2 + b)^{1/2}, \qquad Q = (\tfrac{2}{3}\lambda - ay^2 + cy^3 + 2e^2y^4)^{1/2} \\ F_{12} = iey^2, \qquad F_{23} = F_{24} = F_{34} = 0 \end{gathered} \tag{5.2}$$

where a, b, c, e are arbitrary constants. This metric is conformally flat if and only if $c = e = 0$.

The following solution is symmetrical to the former one:

$$M = 1/x, \qquad P = (\tfrac{2}{3}\lambda - ax^2 + cx^3 - 2e^2x^4)^{1/2}, \qquad Q = (ay^2 + b)^{1/2}$$
$$F_{34} = ex^2, \qquad F_{12} = F_{23} = F_{24} = 0 \tag{5.3}$$

where $a, b, c, e,$ are arbitrary constants. Also this metric is conformally flat if and only if $c = e = 0$.

$$M = 1/2^{1/2}, \qquad P = [(\lambda - e^2 - g^2)x^2 + a]^{1/2}, \qquad Q = [(\lambda + e^2 + g^2)y^2 + b]^{1/2}$$
$$F_{12} = ie, \qquad F_{34} = g, \qquad F_{23} = F_{24} = 0 \tag{5.4}$$

where a, b, e, g are arbitrary constants. This metric is conformally flat if and only if $\lambda = 0$.

$$M = (ay + bx)^{-1}, \qquad P = \left(acx + \tfrac{2}{3}\frac{\lambda}{a^2 + b^2}\right)^{1/2}, \qquad Q = \left(bcy + \tfrac{2}{3}\frac{\lambda}{a^2 + b^2}\right)^{1/2}$$
$$F_{ab} = 0 \tag{5.5}$$

where a, b, c are arbitrary constants. This metric is conformally flat and it includes no electromagnetic field.

The constants appearing in the above solutions are fully arbitrary with the only restriction that they cannot vanish making any of the functions M, P, Q undetermined or equal to zero [cf. (2.9), (3.5) and (3.6)].

Appendix

The proof of Theorem 1, at least in the way it was performed, is too long to allow its presentation here in totality. Therefore, we shall give here only a scheme of the proof, putting emphasis on the more important items so as to allow anybody interested to easily carry out the proof in totality.

First we shall write the general relations necessary for constructing the proof. Let a, b be auxiliary quantities which by (2.2) can be defined as follows:

$$a \underset{\text{def}}{=} \frac{A}{\Omega} \tag{A.1a}$$

$$b \underset{\text{def}}{=} \frac{B}{\Omega} \tag{A.1b}$$

which according to (2.1a) and (2.1b) gives

$$du = ae^1 - be^3 \tag{A.2a}$$

$$dv = ae^2 - be^4 \tag{A.2b}$$

$$w \underset{\text{def}}{=} \tfrac{1}{2}(u - v) \tag{A.3a}$$

$$s \underset{\text{def}}{=} \tfrac{1}{2}(u + v) \tag{A.3b}$$

From (A.2) and (A.3) we obtain

$$du \wedge dv \wedge d\bar{u} \wedge d\bar{v} = 4\, dw \wedge ds \wedge d\bar{w} \wedge d\bar{s} = [(a\bar{b})^2 - (\bar{a}b)^2]\, e^1 \wedge e^2 \wedge e^3 \wedge e^4 \tag{A.4}$$

$$E^1(a\bar{b} - \bar{a}b) = 2(\bar{b}\, ds - b\, d\bar{s}) \tag{A.5a}$$

$$E^2(a\bar{b} + \bar{a}b) = 2i(\bar{b}\, dw - b\, d\bar{w}) \tag{A.5b}$$

$$E^3(a\bar{b} - \bar{a}b) = 2(\bar{a}\, ds - a\, d\bar{s}) \tag{A.5c}$$

$$E^4(a\bar{b} + \bar{a}b) = -2(\bar{a}\, dw + a\, d\bar{w}) \tag{A.5d}$$

The first Cartan formulas, if Γ_{abc} are given by relations (2.1a)-(2.1c), take by (2.8) the form

$$dE^1 = \operatorname{Re} A\, E^1 \wedge E^3 + \operatorname{Im}(A + D)\, E^4 \wedge E^2 \tag{A.6a}$$

$$dE^2 = -(\operatorname{Re} C\, E^1 + \operatorname{Re} A\, E^3) \wedge E^2 + 2 \operatorname{Im} B\, E^4 \wedge E^3 + \operatorname{Im}(A + D)\, E^1 \wedge E^4 \tag{A.6b}$$

$$dE^3 = \operatorname{Re} B\, E^3 \wedge E^1 + \operatorname{Im}(B + C)\, E^2 \wedge E^4 \tag{A.6c}$$

$$dE^4 = -(\operatorname{Re} B\, E^1 + \operatorname{Re} D\, E^3) \wedge E^4 + 2 \operatorname{Im} A\, E^1 \wedge E^2 + \operatorname{Im}(B + C)\, E^2 \wedge E^3 \tag{A.6d}$$

Equations (A.4) and (A.5) suggest that the proof of Theorem 1 should be divided into four separate cases:

$$a\bar{b} + \bar{a}b \neq 0, \qquad a\bar{b} - \bar{a}b \neq 0 \tag{A.7a}$$

$$a\bar{b} + \bar{a}b = 0, \qquad a\bar{b} - \bar{a}b \neq 0 \tag{A.7b}$$

$$a\bar{b} + \bar{a}b \neq 0, \qquad a\bar{b} - \bar{a}b = 0 \tag{A.7c}$$

$$a\bar{b} + \bar{a}b = 0, \qquad a\bar{b} - \bar{a}b = 0 \tag{A.7d}$$

And that is how we shall proceed.

The proof in case (A.7a) is relatively simple, since then from (A.4) we have $dw \wedge ds \wedge d\bar{w} \wedge d\bar{s} \neq 0$ and the functions $w, \bar{w}, s, \bar{s}$ may be used as independent coordinates; moreover all $E_\mu{}^i$ may be calculated from equations (A.5) as functions of $a, \bar{a}, b, \bar{b}$. After these steps are performed we substitute E^i into (A.6a) and (A.6c) and by comparing the coefficients at the same forms $dx^\mu \wedge dx^\nu$ we find that $\operatorname{Im}(A + D) = \operatorname{Im}(B + C) = 0$ and that the functions a, b depend only on the coordinates s and $\bar{s}$. Making use of the latter information, taking account of the theorem of existence of an integrating factor, and after performing adequate partial transformation of the coordinates $s, \bar{s} \to x, y$, from equations (A.5a) and (A.5c) we obtain equations (2.11a) and (2.11c). Subsequently after partial transformation of $w = z + it$ we get from (A.5b) and (A.5d) equations (2.11b) and (2.11d), which in consequence give the proof of Theorem 1 in the case (A.7a).

The proofs in the remaining cases given in (A.7) are more complicated, since we can neither use all the functions $s, \bar{s}, w, \bar{w}$ as independent coordinates nor obtain all E^i forms from equations (A.5).

In case (A.7b), assuming $a = a_0 e^{i\alpha}$, we obtain that $b = \pm ib_0 e^{i\alpha} = ice^{i\alpha}$, where $c =_{\text{def}} \pm b_0$. Subsequently, from equations (A.5b) and (A.5d), whose left-hand sides are equal to zero, we find that $e^{-2i\alpha} dw = -d\bar{w}$, i.e., that $\alpha = \alpha(w)$ and that $e^{-i\alpha} dw$ is imaginary. Denoting $2e^{-i\alpha} dw =_{\text{def}} i\, dz$ we take the real quantity z as a coordinate. From relations (A.2) and (A.3) we then have

$$dz = a_0 E^2 + cE^4 \tag{A.8}$$

Subsequently we introduce the coordinate system $s, \bar{s}, z, r$, where $r = \bar{r}$, and having calculated E^1, E^3 from (A.5a) and (A.5c) we substitute them into equations (A.6a) and (A.6c). We obtain therefrom, in view of the fact that all E^i are independent, that $\text{Im}(A + D) = \text{Im}(B + C) = 0$; hence we conclude further that $\alpha = \text{const}$. The latter allows us to perform the following partial transformation of coordinates: $dx = e^{-i\alpha} ds + e^{i\alpha} d\bar{s}$, $dy = i(e^{-i\alpha} ds - e^{i\alpha} d\bar{s})$ as a result of which we find ourselves in the system of real coordinates x, y, z, r. Making use of this transformation in equations (A.5a) and (A.5c) we find that $E^1 = dx/a_0$ and $E^3 = dv/c$ [of course $a_0, c \neq 0$ by (A.7b)]. Substituting such E^1 and E^3 into (A.6a) and (A.6c) we conclude that the functions a_0, c may depend only on the variables x, y which gives us the relations (2.11a) and (2.11c). In this way the part of Theorem 1 for case (A.7b) has been proved.

To prove the remaining parts of the theorem for case (A.7b) we shall write the forms E^2 and E^4 in their most general shape in the system of coordinates x, y, z, r, i.e., $E^2 = E_\mu{}^2 \cdot dx^\mu$ and $E^4 = E_\mu{}^4 \cdot dx^\mu$. From equation (A.8) it follows by virtue of the fact that the forms E^i span the whole space-time that $E_r{}^4 = -(a_0/c)E_r{}^2 \neq 0$. The fact that $E_r{}^4 \neq 0$ is crucial for the further proof, and we shall frequently make significant use of it. Now our objective is to find the functions $E_\mu{}^2$, and $E_\mu{}^4$, which by (A.8) reduces to determining one of the systems of these quantities, e.g., $E_\mu{}^4$. For this purpose we act with operator d on both sides of equation (A.8), and making use of the relations (A.6b), (A.6d), (A.8) and taking account of $E_r{}^4 \neq 0$, we obtain as a result the system of equations

$$\begin{aligned}
&a_{0,x} - \text{Re}\, C + 2(c/a_0)\, \text{Im}\, A = 0\\
&c_{,x} \quad - (c/a_0)\, \text{Re}\, B = 0\\
&a_{0,y} - (a_0/c)\, \text{Re}\, A = 0\\
&c_{,y} - \text{Re}\, D - 2(a_0/c)\, \text{Im}\, B = 0
\end{aligned} \tag{A.9}$$

Subsequently we take advantage of relation (2.1d) making use of the function $\Omega' =_{\text{def}} \Omega e^{i\alpha}$ instead of Ω which is much more convenient. Putting $\Omega' = Ve^{i\beta}$, we find from equation (2.1d), since $E^1 \sim dx$ and $E^3 \sim dy$, that V and β may be functions of x, y only. Relation (2.1d), considering that $\text{Im}(A + D) =$

Im$(B + C) = 0$, gives the following system of equations:

$$\begin{aligned}
\partial_x \ln V &= -(1/a_0)\,\mathrm{Re}\, C \\
\partial_y \ln V &= -(1/c)\,\mathrm{Re}\, D \\
\beta_{,x} &= (1/a_0)\,\mathrm{Im}\, B \\
\beta_{,y} &= (1/c)\,\mathrm{Im}\, A
\end{aligned} \tag{A.10}$$

Combining equations (A.9) and (A.10), and making use of (A.1) we obtain the conditions that $\beta_{,x}, \beta_{,y} \neq 0$ and an easily integrable system of differential equations from which we find a_0, c, V, β, as simple functions of the variable x, y and of disposable but different from zero functions $m(x)$ and $n(y)$ [after performing transformations $x = x(x')$, $y = y(y')$ immaterial as regards the shape of relations (2.11a) and (2.11c), and dropping the primes] which gives us from (A.1) and (A.10) expressions for a_0, c, and for the real and imaginary parts of A, B, C, D. Substituting these expressions into (A.6d) we obtain the following system of differential equations:

$$\begin{aligned}
E^4_{y,x} - E^4_{x,y} &= -\frac{xy^2}{1 + x^2y^2} E_y{}^4 + \left[\frac{n_{,y}}{n} - \frac{2 + x^2y^2}{y(1 + x^2y^2)}\right] E_x{}^4 \\
E^4_{z,x} - E^4_{x,z} &= -\frac{xy^2}{1 + x^2y^2} E_z{}^4 + \frac{2xy^2}{n(1 + x^2y^2)^{1/2}} \\
E^4_{r,x} - E^4_{x,r} &= -\frac{xy^2}{1 + x^2y^2} E_r{}^4 \\
E^4_{z,y} - E^4_{y,z} &= \left[\frac{2 + x^2y^2}{y(1 + x^2y^2)} - \frac{n_{,y}}{n}\right] E_z{}^4 \\
E^4_{r,y} - E^4_{y,r} &= \left[\frac{2 + x^2y^2}{y(1 + x^2y^2)} - \frac{n_{,y}}{n}\right] E_r{}^4 \\
E^4_{r,z} - E^4_{z,r} &= 0
\end{aligned} \tag{A.11}$$

Here equation (A.6b) results from equations (A.6d) and (A.8) if we take account of (A.9), and therefore it does not give any new relations. Integration of equations (A.11) becomes easier if we introduce new functions W, S as follows: $W_{,z} = E_z{}^4$, $S_{,r} = E_r{}^4$. The integration itself is computationally cumbersome because of the number of equations in the system (A.11) and because of the number of disposable functions of the variables x, y, z, r that appear. Making use of the condition $E_r{}^4 \neq 0$ and performing in the course of integration a transformation of coordinate system of the type

$$x, y, z, r \to x, y, z, h(x, y, z, r, T_1, T_2, \ldots) \tag{A.12}$$

where T_i are the mentioned disposable functions appearing in the course of integration, we finally find such a transformation $h = t$ of (A.12) type that in the coordinate system x, y, z, t we have $E_x{}^4 = E_y{}^4 = 0$, $E_z{}^4(x, y)$, $E_t{}^4(x, y) \neq 0$, from which by (A.8) we obtain relations (2.11b) and (2.11d), which terminates the proof of Theorem 1 for the case (A.7b).

In the case (A.7c), assuming $a = a_0 e^{i\alpha}$, we obtain $b = \pm b_0 e^{i\alpha} = ce^{i\alpha}$, where $c =_{\text{def}} \pm b_0$. Subsequently from equations (A.5a) and (A.5c), whose left-hand sides are now equal to zero, we find that $e^{-2i\alpha} ds = d\bar{s}$. Hence $\alpha = \alpha(s)$, and so, $e^{-i\alpha} ds$ is real. Denoting $2e^{-i\alpha} ds = dy$ we assume the real quantity y as coordinate. From relations (A.2) and (A.3) we then obtain

$$dy = a_0 E^1 - cE^3 \tag{A.13}$$

If we introduce the system of coordinates $x, y, w, \bar{w}$, where $x = \bar{x}$, then from equations (A.5b) and (A.5d) we find that $E_x{}^2 = E_y{}^2 = E_x{}^4 = E_y{}^4 = 0$. Since all E^i forms span the whole space-time, from the latter equations and (A.13) we get $E_x{}^1 = (c/a_0) \cdot E_x{}^3 \neq 0$. The fact that $E_x{}^1 \neq 0$ is crucial for conducting further proof, as we shall frequently make significant use of it. Subsequently we calculate E^2 and E^4 from relations (A.5b), (A.5d) and substitute them into equations (A.6b) and (A.6d). Comparing the coefficients at different forms $dx^\mu \wedge dx^\nu$ we obtain a system of separate equations from the analysis of which it follows readily that $\operatorname{Im} A = \operatorname{Im} B = \operatorname{Im} C = \operatorname{Im} D = 0$ and that $\alpha = \text{const}$ if $E_x{}^1 \neq 0$, which as we know holds. Hence the following partial transformation of coordinates is possible, viz., $i(e^{-i\alpha} dw - e^{i\alpha} d\bar{w}) = dz$, $-(e^{-i\alpha} dw + e^{i\alpha} \cdot d\bar{w}) = dt$, after realization of which we obtain from relations (A.5b) and (A.5d):

$$E^2 = dz/a_0, \qquad E^4 = dt/c \tag{A.14}$$

In this way we have found ourselves in the system of real coordinates x, y, z, t. Now let $\Omega' =_{\text{def}} \Omega e^{i\alpha}$. In the following we shall be using the function Ω' instead of Ω which is by far more convenient. Since $\operatorname{Im} A = \operatorname{Im} B = 0$, from (A.1) we get $A = \bar{A} = a_0 \Omega'$, $B = \bar{B} = c\Omega'$, and thus Ω' is real. Hence in case (A.7c) we have $\operatorname{Im} A = \operatorname{Im} B = \operatorname{Im} C = \operatorname{Im} D = \operatorname{Im} \Omega' = 0$, and therefore in the following part of the proof we shall be able to use only real quantities. In our new system of coordinates x, y, z, t the most general shape of forms E^1, E^3 is as follows: $E^1 = E_\mu{}^1 \cdot dx^\mu$, $E^3 = E_\mu{}^3 \cdot dx^\mu$, the relation $E_x{}^1, E_x{}^3 \neq 0$ still holding since the partial transformation of coordinates took place between $w, \bar{w}$ and z, t, only. Assuming such shapes of forms E^1 and E^3 we obtain, after acting on both sides of equation (A.13) with operator d and making use of relations (A.6a) and (A.13) and condition $\operatorname{Im}(A + D) = 0$, the system of equations

$$E_\mu{}^1 \partial_\nu \frac{a_0}{c} - E_\nu{}^1 \partial_\mu \frac{a_0}{c} + \left[\frac{c_{,\nu}}{c^2} - \Omega' E_\nu{}^1 \left(1 + \frac{a_0{}^2}{c^2}\right)\right] \delta_\mu{}^y = 0 \tag{A.15}$$

where $\mu, \nu = x, y, z, t$ and $\mu \neq \nu$. From relations (A.6b), (A.13) and (A.6d), (A.13) we obtain two systems of equations, alternately:

$$a_{0,\mu} = \frac{a_0}{c}(cC + a_0{}^2\Omega')E_\mu{}^1 - \frac{a_0{}^2}{c}\Omega'\delta_\mu{}^y, \qquad \mu = x, y, t \tag{A.16}$$

$$c,_\mu = (Da_0 + c^2\Omega')E_\mu{}^1 - D\delta_\mu{}^y, \qquad \mu = x, y, z \tag{A.17}$$

Let us denote $\gamma =_{\text{def}} (cC + a_0D) = (1/\Omega')(AD + BC)$. The remaining part of the proof splits into two cases: $\gamma \neq 0$ and $\gamma = 0$.

If $\gamma \neq 0$, then from (2.1d) and (A.13) we get $E^1 = (1/\gamma)(D\,dy - cd \ln \Omega')$. Since $E_x{}^1 \neq 0$ we have $\Omega'_{,x} \neq 0$ and we can make the following transformation of coordinates: $x, y, z, t \to X, y, z, t$, where $X = \ln \Omega'$. Substituting the forms E^1, E^3 expressed in the new system of coordinates into (A.6a), (A.6c) and analyzing the coefficients at independent forms $dx^\mu \wedge dx^\nu$ we find that $a_0,_t = a_0,_z = c,_z = c,_t = 0$ and that $E^1 = F_1(X, y)\,dX + F_2(X, y)\,dy$, $E^3 = F_3(X, y)\,dX + F_4(X, y)\,dy$, from which, after making use of the integrating factor existence theorem, performing an adequate partial transformation of the coordinates $x'(X, y), y'(X, y)$, and dropping the primes, we obtain by (A.14) a particular case of equations (2.11) when $l = p = 0$.

If on the other hand $\gamma = 0$, then we must have $cC + a_0{}^2\Omega' \neq 0$ or $c^2\Omega' + a_0D \neq 0$, since if both these expressions were equal to zero, then because of $\gamma = 0$ we would have to have $\Omega' = 0$ or $a_0{}^2 + c^2 = 0$, which is contradictory to the assumptions. Now the proof for the case considered splits into two further branches, viz., when $(a_0/c),_x = 0$ and when $(a_0/c),_x \neq 0$. If $(a_0/c),_x = 0$, then from equations (A.16) and (A.17) and considering $E_x{}^1 \neq 0$ we find that $\sigma =_{\text{def}} cC + a_0{}^2\Omega' = c^2\Omega' + a_0D \neq 0$, wherefrom it follows that $\partial_x \ln a_0 = \partial_x \ln c \neq 0$. After transformation of the coordinates: $x, y, z, t \to X, y, z, t$, where, e.g., $X = \ln c$ (or possibly $X = \ln a_0$) we obtain $E^1 = 1/\sigma(c\,dX + D\,dy)$, the continuation of the proof being the same as in the case $\gamma \neq 0$ starting from an analogous point. If on the other hand $(a_0/c),_x \neq 0$, then we consider separately four variants, i.e., all pairs when $E_z{}^1, E_t{}^1$ are equal to or different from zero. After analyzing by means of equations (A.6a), (A.13), (A.15), (A.16), and (A.17) each of the three variants different from $E_z{}^1 = E_t{}^1 = 0$ (applying where necessary transformations of coordinates analogous to those described in the preceding situations), we reach the conclusion that they are contradictory. Thus only the variant $E_z{}^1 = E_t{}^1 = E_t{}^3 = E_z{}^3 = 0$ remains. Here we obtain immediately from equations (A.15)–(A.17) that $a_{0,z} = a_0,_t = c,_z = c,_t = 0$, and hence from equations (A.6a), (A.13) we conclude that the functions $E_x{}^1, E_y{}^1, E_x{}^3, E_y{}^3$ depend only on the variables x, y. After applying the integrating factor existence theorem, adequate partial transformation of coordinates $x'(x, y)$, $y'(x, y)$ and dropping the primes, by relations (A.14) we get equations (2.11) for $l = p = 0$, which terminates the proof of Theorem 1 in case (A.7c).

In case (A.7d) at least one of the quantities a, b must be equal to zero. Thus the proof splits into three separate subcases: (1) $a \neq 0, b = 0$; (2) $a = 0$, $b \neq 0$; (3) $a = b = 0$.

In subcase (1) let us assume that $a = a_0 e^{i\alpha}$. From equations (A.2), (A.3), (A.5c), and (A.5d) we then obtain that the expressions $e^{-i\alpha}\, ds$ and $e^{-i\alpha}\, dw$ are real and imaginary, respectively, and that α = const. In this situation we can introduce two real coordinates x, z in the following manner: $dx = 2e^{-i\alpha}\, ds$, $dz = 2ie^{-i\alpha}\, dw$. From this we have $E^1 = dx/a_0$, $E^2 = dz/a_0$. Let m, n be the other two real coordinates. The most general shapes of the forms E^3 and E^4 in the coordinate system x, z, m, n are $E^3 = E_\mu{}^3 \cdot dx^\mu$, $E^4 = E_\mu{}^4 \cdot dx^\mu$. Substituting these E^i into (A.6a) and analyzing the coefficients at different forms $dx^\mu \wedge dx^\nu$ by virtue of the fact that E^i span the whole space-time we find that $\mathrm{Im}(A + D) = 0$. Let $\Omega' =_{\mathrm{def}} \Omega e^{i\alpha}$. We then get from (A.1a) that $A = a_0 \Omega'$. Let us now assume that $\mathrm{Re}\, A = 0$ (which means that $\mathrm{Re}\, \Omega' = 0$). Analyzing the imaginary part of equation (2.1d) we obtain, considering the facts that E^i span the whole space-time and that $\mathrm{Im}(A + D) = 0$, the following relation: $\mathrm{Im}\, A = 0$. This is contradictory to the assumption that $a \neq 0$. Hence $\mathrm{Re}\, A \neq 0$. We then obtain from relations (A.6a) and (A.6b) the following system of equations:

$$E_\mu{}^3 = \frac{1}{a_0\, \mathrm{Re}\, A} (\partial_\mu a_0 - \delta_\mu{}^x\, \mathrm{Re}\, C), \qquad \mu = x, z, m, n \qquad \text{(A.18)}$$

Since $E^1 \sim dx$, $E^2 \sim dz$, we must have $E_m{}^3 \neq 0$ or $E_n{}^3 \neq 0$, and hence by (A.18) $a_{0,m} \neq 0$ or $a_{0,n} \neq 0$. Let, e.g., $a_{0,m} \neq 0$. We shall transform the coordinate system $x, z, m, n \to x, z, r, n$, where $r = \ln a_0$. From (A.18) we then find that $E^3 = (1/\mathrm{Re}\, A)\,(dr - e^{-r}\, \mathrm{Re}\, C\, dx)$, and in the new coordinate system we have $E^4 = E_x{}^4\, dx + E_z{}^4\, dz + E_r{}^4\, dr + E_n{}^4\, dn$, where we must have $E_n{}^4 \neq 0$. Substituting E^i expressed in this system of coordinates into (A.6c) we find, considering $E_n{}^4 \neq 0$, that $\mathrm{Im}\, C = 0$, i.e., $dE^3 = 0$. Thus there exists such a quantity y that $E^3 = dy$. After transformation of the coordinates $x, r, z, n \to x, y, z, n$ we get

$$E^1 = \frac{dx}{a_0}, \qquad E^2 = \frac{dz}{a_0}, \qquad E^3 = dy \qquad \text{(A.19)}$$

and it is easy to show, making use among other things of equation (2.1d), that $a_0 = a_0(x, y)$. Thus we have obtained relations (2.11a)-(2.11c). The form E^4 is still to be found. With this in view let us assume that $\Omega' = Ve^{i\beta}$. From relation (2.1d) and since $C = \overline{C}$ we get that $\beta = \beta(y)$, $\mathrm{Im}\, A = \beta_{,y}$. From the latter two relations and from equations $A = \Omega' a_0$ as well as (2.1d), (A.6b), (A.19) it follows that each of the $a_0(x, y)$ and $V(x, y)$ functions must be the product of two functions of one variable, and their product $a_0 V$ is a function of y only. Hence we see that $\mathrm{Re}\, D = -\partial_y \ln V$ is also a function of y only. Making use of these relations and of equations (A.19) in (A.6d) we get, in view of the fact that $B = \mathrm{Im}\, C = 0$, an easy to analyze relation determining the form E^4. It is most convenient to split this analysis into four separate cases of all combinations when the quantities $\mathrm{Im}\, A$ and $\mathrm{Re}\, D$ are equal to or different from zero. In three variants when at least one of the quantities

Im A or Re D is equal to zero we obtain easily from the Darboux theorems, either directly or making additionally use of the fact that $d\,dE^4 = 0$, certain cases of equation (2.11d). In the situation when Im $A \neq 0$ and Re $D \neq 0$ we can make the following transformation of the coordinates $x(x')$, $y(y')$: $-\text{Re}\, D\, dy = dy'$, $2(\text{Im}\, A/a_0{}^2) \cdot dx = e^{y'}\, dx'$, as a result of which equation (A.6d) will take the form $dE^4 = dy' \wedge E^4 + e^{y'}\, dx' \wedge dz$. Integrating this equation and simultaneously, where possible, making the transformation $dx^\mu = \Sigma_\nu\, dx^\nu$, we finally get a new coordinate t such that $E^4 = e^{y'}(dt + x'\, dz)$. If we make a reciprocal transformation of the coordinates $x'(x)$, $y'(y)$ or drop the primes, we get a particular case of relation (2.11d), which together with (A.19) proves Theorem 1 for the variant considered.

The proof of Theorem 1 in case (A.7d) and subcase (2), when $a = 0$ and $b \neq 0$ is fully analogous to subcase (1), as it suffices to make simultaneous mutual displacements $A \leftrightarrow B$, $C \leftrightarrow D$, $E^1 \leftrightarrow E^3$, $E^2 \leftrightarrow E^4$.

The proof for subcase (3), when $a = b = 0$, is simple if we use equations (2.1e) as an independent assumption, which is made here for the first time in the proof of Theorem 1. From (2.1e) by (A.6a) and (A.6c) we obtain that $E^1 = dx$, $E^3 = dy$, and by (A.6b), (A.6d) and the Darboux theorems we get that $E^2 = \sigma\, dz'$, $E^4 = \rho\, dt'$. Substituting such E^i into (A.6b), (A.6d) and taking into account that $C = C(x, y)$, $D = D(x, y)$ result from (2.1d), we find that $C(x)$, $D(y)$, $\sigma = 2k(x)\sigma'(z')$, $\rho = 2q(y) \cdot \rho'(t')$, which after the transformation $dz = \sigma'\, dz'$, $dt = \rho'\, dt'$, yields a subcase of relations (2.11).

In this way the proof of Theorem 1 has been terminated.

It deserves emphasizing that we have used assumption (2.1e) as an independent assumption only in the proof of subcase (3) ($a = b = 0$) of case (A.7d). In all the remaining cases relation (2.1e) has been obtained as a conclusion from assumptions (2.1a)-(2.1d). From the formal point of view Theorem 1 can be proved without using (2.1e) as assumption, if instead of it we use the assumption that $A \neq 0$ or $B \neq 0$. In that case relation (2.1e) will always be a conclusion from thus modified assumptions (2.1), but the class of metrics (2.11) determined by these assumptions would be poorer as it would not include cases for which $A = B = 0$.

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